MOTION IN A PLANE.

INTRODUCTION

In order to describe motion of an object in two dimensions (a plane) or three dimensions (space), we need to use vectors to describe the concepts of position, displacement, velocity and acceleration. Therefore, it is first necessary to learn the language of vectors. What is a vector? How to add, subtract and multiply vectors? What is the result of multiplying a vector by a real number?

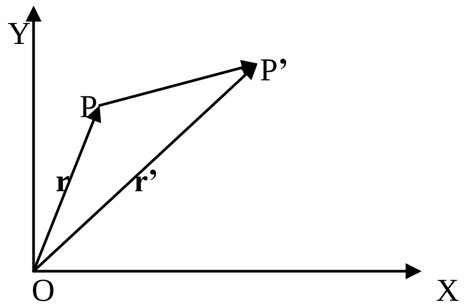
SCALARS AND VECTORS

A scalar quantity is a quantity with magnitude only. It is specified completely by a single number, along with the proper unit. Examples are : distance, mass, the temperature and the time. The rules for combining scalars are the rules of ordinary algebra. Scalars can be added, subtracted, multiplied and divided just as the ordinary numbers.

A vector quantity is a quantity that has both a magnitude and a direction. A vector is specified by giving its magnitude by a number and its direction. Example: displacement, velocity, acceleration and force.

Vector is represented by an arrow placed over a letter, say $\vec{v}$.

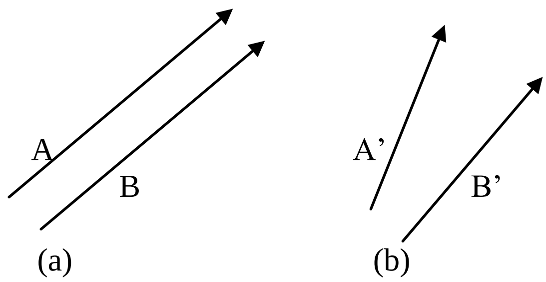
Position and Displacement Vectors



Let P and P' be the positions of the object at time t and t'. We join O and P by a straight line. Then, **OP** is the position vector of the object at time t. i.e. **OP** = **r**. Similarly **OP'** denoted by **r'** If the object moves from P to P' , the vector **PP'** (with tail at P and tip at P') is called the **displacement vector** corresponding to motion from point P (at time t) to point P' (at time t').

Equality of Vectors

Two vectors **A** and **B** are said to be equal if, and only if, they have the same magnitude and the same direction.



(a) Two equal vectors **A** and **B**. (b) Two vectors **A'** and **B'** are unequal though they are of the same length

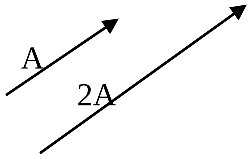
MULTIPLICATION OF VECTORS BY REAL NUMBERS

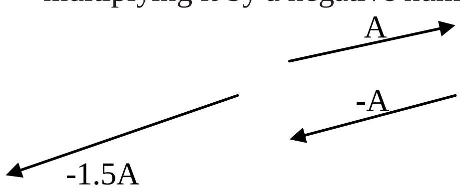
Multiplying a vector **A** with a positive number λ gives a vector whose magnitude is changed by the factor λ but the direction is the same as that of **A** :

$$|\lambda \mathbf{A}|=\lambda\ |\mathbf{A}| \text{ if } \lambda > 0.$$

a) Vector **A** and the resultant vector after multiplying **A** by a positive number 2.

(b) Vector **A** and resultant vectors after multiplying it by a negative number –1and –1.5

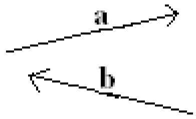




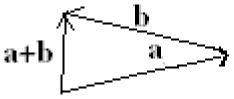
For example, if we multiply a constant velocity vector by duration (of time), we get a displacement vector.

ADDITION AND SUBTRACTION OF VECTORS — GRAPHICAL METHOD

Triangle method of vector addition



For adding any two vectors **a** and **b** in a plane and get resultant **a + b**, we shift **b** parallel to itself until its tail coincides with the tip of **a.** Then complete the triangle to get **a + b** as shown in the figure.

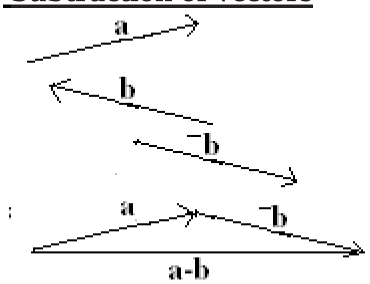


Vector addition is **commutative**: **A** + **B** = **B** + **A**
The result of adding vectors **A** and **B** first and then adding vector **C** is the same as the result of adding **B** and **C** first and then adding vector **A** : (**A** + **B**) + **C** = **A** + (**B** + **C**)
Hence the addition of vectors also obeys the **associative** law.

Null vector or a **zero vector** :

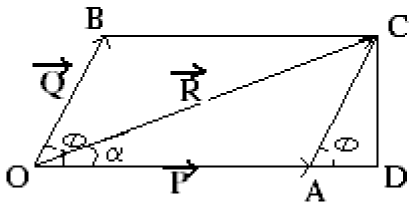
A vector that has zero magnitude and is represented by **0** called a **null vector** or a **zero vector** :
Eg: **A** – **A** = **0** |**0**|= 0 Since the magnitude of a null vector is zero, its direction cannot be specified.
The main properties of **0** are : **A** + **0** = **A,** λ **0** = **0,** 0 **A** = **0.**
Physical eg: An object moves from a point and come back to the same point, then its displacement is a Null vector.

Subtraction of vectors



To subtract **b** from **a,** first find **–b** then add to **a** to get '**a – b'** as shown in the figure.
Mathematically, **a – b = a + (-b) = a + (-1) b**

Parallelogram law of addition: -
The law states that if two vectors acting at a point are represented in magnitude and direction by the adjacent sides of a parallelogram, then the resultant is given by the diagonal passing through the point.
Here **P & Q** are two vectors acting at O, then the resultant **R** of two vectors is given by the diagonal OC.



Here magnitude of resultant,

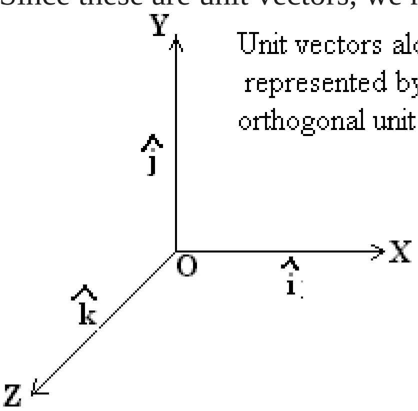
$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

Direction, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

Rain is falling vertically with a speed of 35 m s^{-1}. Wind starts blowing after sometime with a speed of 12 m s^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella?

Unit vectors: -

It is a vector of unit magnitude drawn in the direction of any given vector.
Let **A** be a vector then the unit vector along its direction is written as A or $\hat{\mathbf{a}}$.
Magnitude of the vector $/\mathbf{A}/ = \dfrac{\mathbf{A}}{\mathbf{a}}$, ∴Unit vector $\mathbf{a} = \dfrac{\mathbf{A}}{/\mathbf{A}/}$
Since these are unit vectors, we have | ˆ**i** |= | ˆ**j**| = | ˆ**k**| =1



Unit vectors along X ,Y, Z directions are represented by $\hat{i}$, $\hat{j}$, & $\hat{k}$ and these are called orthogonal unit vectors

RESOLUTION OF VECTORS

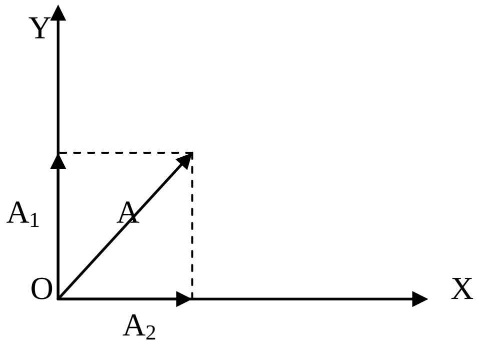
The splitting up of a single vector into a number of vectors is called resolution of a vector. Each vector thus obtained is known as component vectors.

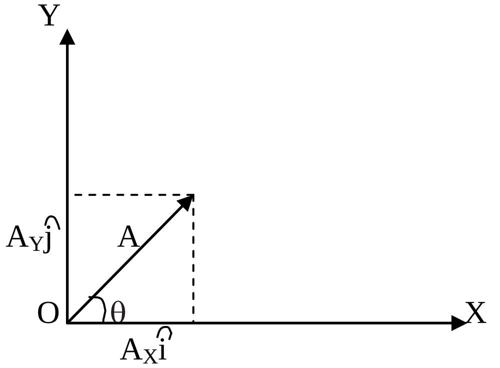
Rectangular component of a vector in a plane: -

The resolution of a vector into two mutually perpendicular vectors is called the rectangular resolution of the vector.

We can now resolve a vector **A** in terms of component vectors that lie along unit vectors **i**ˆ and **j**ˆ . Consider a vector **A** that lies in *x-y* plane as shown in Fig. We draw lines from the head of **A** perpendicular to the coordinate axes as in Fig and get vectors $\mathbf{A_1}$ and $\mathbf{A_2}$ such that $\mathbf{A_1} + \mathbf{A_2} = \mathbf{A}$. Since $\mathbf{A_1}$ is parallel to **i**ˆ and $\mathbf{A_2}$ is parallel to **j**ˆ, we have : $\mathbf{A_1} = A_x\,\hat{\mathbf{i}}$, $\mathbf{A2} = A_y\,\hat{\mathbf{j}}$ where *Ax* and *Ay* are real numbers.

Thus, $\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$





The quantities *Ax* and *Ay* are called *x*-, and *y*- components of the vector **A**.

From figure, we have $A_x = A\cos\theta$, and $A_y = A\sin\theta$

ie $A_x^{\,2} + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta$

$= A^2(\cos^2\theta + \sin^2\theta) = A^2$

Or, $A = \sqrt{A_x^2 + A_y^2}$ and $\tan\theta = \dfrac{A_y}{A_x}$ or $\theta = \tan^{-}\dfrac{A_y}{A_x}$

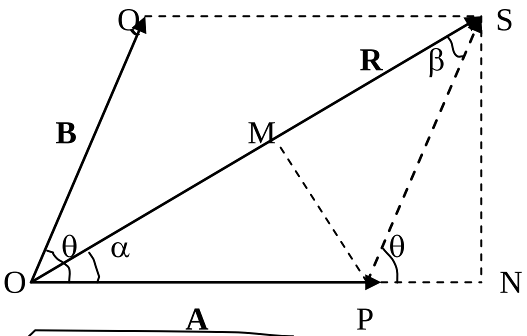
A vector **A** resolved into components along x-, y-, and z-axes; $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$

The magnitude of vector **A** is $A = \sqrt{A_x^{\,2} + A_y^2 + A_z^2}$

VECTOR ADDITION – ANALYTICAL METHOD

Consider two vectors **A** and **B** in x-y plane with components Ax, Ay and Bx, By : Find their resultant R in component form.

Find the magnitude and direction of the resultant of two vectors **A** and **B** in terms of their magnitudes and angle θ between them.



From figure prove that $\mathbf{R} = \sqrt{\mathbf{A^2 + B^2 + 2AB\cos\theta}}$ This Equation) is known as the **Law of cosines** and Eq. $\dfrac{R}{\sin\theta} = \dfrac{A}{\sin\beta} = \dfrac{B}{\sin\alpha}$ as the **Law of sines**.

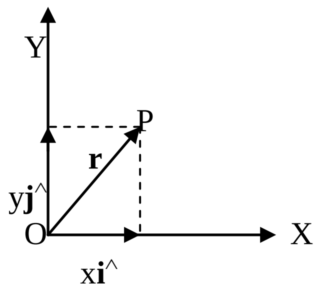
The direction of resultant is given by $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$

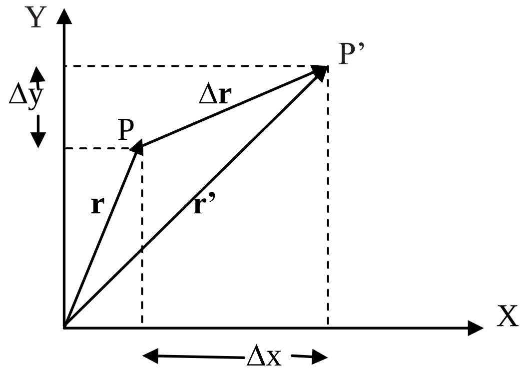
A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

MOTION IN A PLANE

Position Vector and Displacement

The position vector **r** of a particle P located in a plane with reference to the origin of an *x-y* reference frame is given by $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\boldsymbol{j}}$ where *x* and *y* are components of **r** along *x*-, and *y*- axes.





Let a particle is at P at time t and P′ at time t'. Then, the displacement is : $\Delta \mathbf{r} = \mathbf{r}' - \mathbf{r}$ and is directed from P to P′.

We can write above Eq. in a component form:

$$\Delta \mathbf{r} = (x' \mathbf{i} + y' \mathbf{j}) - (x \mathbf{i} + y \mathbf{j})$$

$$= \mathbf{i}\Delta x + \mathbf{j}\Delta y \quad \text{where } \Delta x = x' - x,\ \Delta y = y' - y$$

Velocity

The average velocity $(\bar{\mathbf{v}})$ of an object is the ratio of the displacement and the corresponding time interval:

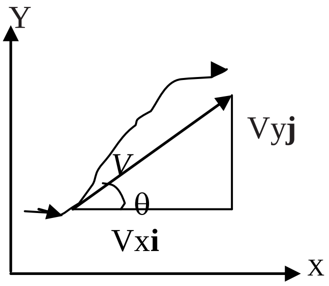
$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}}{\Delta t} = \mathbf{i}\frac{\Delta x}{\Delta t} + \mathbf{j}\frac{\Delta y}{\Delta t}$$

Or $\bar{\mathbf{v}} = \bar{v}x\,\mathbf{i} + \bar{v}y\,\mathbf{j}$

Since $\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t}$ the direction of the average velocity is the same as that of $\Delta \mathbf{r}$

The **velocity** (instantaneous velocity) is given by $\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$

The direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion.



We can express $\mathbf{v}$ in a component form : $\mathbf{v} = \frac{d\mathbf{r}}{dt}$

$$\mathbf{v} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} = v_x \mathbf{i} + v_y \mathbf{j}$$

where $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$

The magnitude of $\mathbf{v}$ is then $v = \sqrt{v_x^2 + v_y^2}$

and the direction of $\mathbf{v}$ is given by the angle θ : $\tan\theta = \frac{v_y}{v_x}$ Or $\theta = \tan^{-1}\left\{\frac{v_y}{v_x}\right\}$

Acceleration

The **average acceleration a** of an object for a time interval Δt moving in x-y plane is the change in velocity divided by the time interval

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta(v_x \mathbf{i} + v_y \mathbf{j})}{\Delta t} = \frac{\Delta v_x}{\Delta t}\mathbf{i} + \frac{\Delta v_y}{\Delta t}\mathbf{j}$$

Or, $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$

The **acceleration** (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero :

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$$

Note that in one dimension, the velocity and the acceleration of an object are always along the same straight line However, for motion in two or three dimensions, velocity and acceleration vectors may have any angle between 0° and 180° between them.

The position of a particle is given by **r** = 3.0t ˆ**i** + 2.0t^2ˆ**j** + 5.0**k**ˆ where t is in seconds and the coefficients have the proper units for **r** to be in metres. (a) Find **v**(t) and **a**(t) of the particle. (b) Find the magnitude and direction of **v**(t) at t = 1.0 s.

MOTION IN A PLANE WITH CONSTANT ACCELERATION

The equations of motions in a plane are as same as equations motion in straight line, but in vector form.

1. $\mathbf{v} = \mathbf{v_0} + \mathbf{a}t$

   In terms of components :

   $v_x = v_{0x} + a_x t$

   $v_y = v_{0y} + a_y t$

2. $\mathbf{r} = \mathbf{r_0} + \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2$

Above equation can be written in component form as

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

That is, **motion in a plane (two-dimensions) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions**.

A particle starts from origin at t = 0 with a velocity 5.0 **î** m/s and moves in x-y plane under action of a force which produces a constant acceleration of (3.0**i** + 2.0**j**) m/s^2. (a) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m ? (b) What is the speed of the particle at this time ?

RELATIVE VELOCITY IN TWO DIMENSIONS

Suppose that two objects A and B are moving with velocities $\mathbf{v_A}$ and $\mathbf{v_B}$. Then, velocity of object A **relative to that of B is** : $\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$

and similarly, the velocity of object B *relative to that of* A is :

$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$

Therefore, $\mathbf{v}_{AB} = -\mathbf{v}_{BA}$ and, $|\mathbf{v}_{AB}| = |\mathbf{v}_{BA}|$

Rain is falling vertically with a speed of 35 m s^{-1}. A woman rides a bicycle with a speed of 12 m s^{-1} in east to west direction. What is the direction in which she should hold her umbrella?

PROJECTILE MOTION

Projectile motion is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.

Projectile : - An object that is given an initial velocity and moves under gravity is called a projectile.

Trajectory : - The path of the projectile is called trajectory. It is a parabola with its axis parallel to the direction of acceleration due to gravity 'g'.

Angle of projection(θ) : - It is the angle between direction of projection and horizontal.

Let a body be projected with an initial velocity Vo. The velocity vector makes an angle θ with the horizontal. Vo can be resolved into two components ; Vox along X-axis and Voy along Y-axis.

Here Vox = Vo cosθ; and Voy = Vo sinθ

Horizontal component Vox is constant throughout the motion and Voy varies due to acceleration due to gravity 'g'.

Let $x_0 = 0$ and $y_0 = 0$ is the initial position of the projectile. After ' t'sec its position be $x = V_{ox} t$

Or x = Vo cosθ t and $y = (v_o \sin\theta)\, t - (\tfrac{1}{2})g\, t^2$

The components of velocity at time t can be obtained using Eq:

$$v_x = v_{ox} = v_o \cos\theta$$

$$v_y = v_o \sin\theta - g\, t$$

Equation of path of a projectile

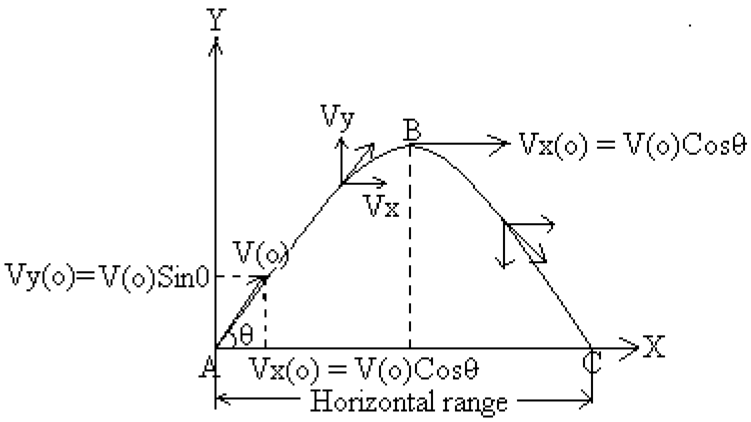
What is the shape of the path followed by the projectile?

Using the equation $y = (v_o \sin\theta)\,t - (\tfrac{1}{2})g\,t^2$

Substituting the value of t from eqn $x = V_o \cos\theta\, t$; we have $t = \dfrac{x}{V_o \cos\theta}$.

$$y = \frac{(v_o \sin\theta)\,x}{V_o \cos\theta} - \frac{g\,(x^2)}{2(V_o \cos\theta)^2}$$

Now, since g, θ and v_o are constants, then above Eq. is of the form $y = a\,x + b\,x^2$, in which a and b are constants. This is the equation of a parabola, i.e. the path of the projectile is a parabola.



Time of flight (T) : -

The time taken to complete the trajectory is called time of flight.

Consider the motion of the projectile from A to C.

Vertical displacement $y(t) - y(o) = 0$; $a_y = -g$; $V_y(o) = V(o)\sin\theta$; Substituting these values in eqn;

$y(t) - y(o) = V_y(o)\,t + \tfrac{1}{2}a\,t^2$

$0 = V(o)\sin\theta\, T - \tfrac{1}{2}g\,T^2$

$V(o)\sin\theta\, T = \tfrac{1}{2}g\,T^2$

Or $$\mathbf{T = \frac{2\,V(o)\sin\theta}{g}}$$

Horizontal range ® : -

The horizontal displacement of the projectile is called horizontal range ®.

Throughout the motion horizontal velocity of the projectile is constant. i.e $V_x(o) = V(o)\cos\theta$ is constant

∴ Horizontal range = Uniform velocity × Time.

$$R = V(o)\cos\theta \times T$$

$$= V(o)\cos\theta \times \frac{2\,V(o)\sin\theta}{g}$$

$$R = \frac{V(o)^2\, 2\sin\theta\cos\theta}{g} \qquad \{2\sin\theta\cos\theta = \sin 2\theta\}$$

$$\mathbf{R = \frac{V(o)^2 \sin 2\theta}{g}}$$

For a given velocity of projection R is maximum when $\sin 2\theta$ is maximum; I.e $\sin 2\theta = 1$;

Or $2\theta = 90^0$ i.e $\theta = 45^0$ ∴ $\mathbf{R_{max} = \frac{V(o)^2}{g}}$

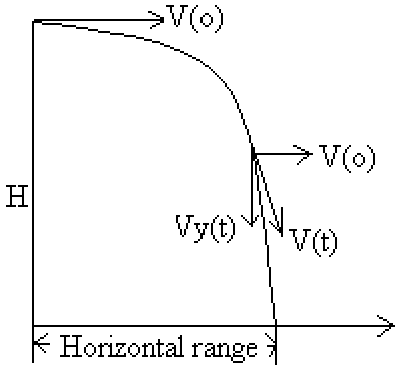
Maximum height (H) : -

The maximum vertical height attained by the projectile is called maximum height.

Here $y(t) – y(o) = H$; $Vy(o) = V(o) \sin\theta$; $Vy(t) = 0$; $a_y = -g$.
Using 3rd equation of motion;
$Vy(t)^2 = Vy(o)^2 + 2a [y(t) – y(o)]$
$0 = V(o)^2 \sin^2\theta - 2 g H$.
$2 g H = V(o)^2 \sin^2\theta$
Or $\mathbf{H = \frac{V(o)^2 \sin^2\theta}{2g}}$

H is maximum when $\sin\theta = 1$; or $\theta = 90^0$ $\therefore \mathbf{Hmax = \frac{V(o)^2}{2g}}$

Body projected horizontally from a height : -



Let the projectile be thrown horizontally with velocity V(o). It will be acted upon by constant acceleration 'g' vertically downward.
Here $Vx(t) = V(o)$ [since horizontal velocity is constant]
$Vy(t) = Vy(o) + at$
$Vy(t) = 0 + gt$ [since there is no vertical component at t = 0; $Vy(o) = 0$]

Let H be the vertical displacement when projectile reaches the ground and T be the time taken to reach the ground.
Then $y(t) – y(o) = Vy(o)\, t + ½\, at^2$
$H = 0 + ½\, g T^2$
$T^2 = \frac{2H}{g}$ or $\mathbf{T = \frac{\sqrt{2H}}{\sqrt{g}}}$

During this interval of time, horizontal distance covered is given by, Range = uniform velocity × time
$\mathbf{R = V(o) \times \frac{\sqrt{2H}}{\sqrt{g}}}$

A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 m s^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 m s^{-2}).

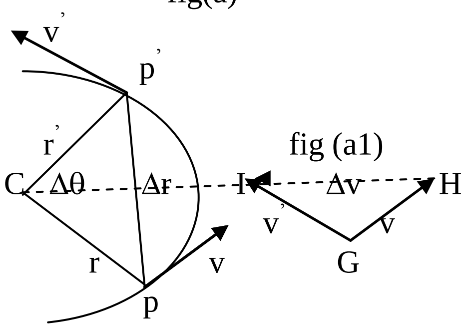
A cricket ball is thrown at a speed of 28 m $s^{–1}$ in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

UNIFORM CIRCULAR MOTION

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion**.
Suppose an object is moving with uniform speed v in a circle of radius R as shown in Fig.(a). Since the velocity of the object is changing continuously in direction, the object undergoes acceleration
Let **r** and $\mathbf{r}'$ be the position vectors and **v** and v^1 the velocities of the object when it is at point P and P' as shown in Fig. (a) . The velocity vectors **v** and $\mathbf{v}'$ are as shown in Fig. (a1).
$\Delta\mathbf{v}$ is obtained in Fig. (a1) using the triangle law of vector addition.
Since the path is circular **v** is perpendicular to **r** and so is $\mathbf{v}'$ to $\mathbf{r}'$
Therefore, $\Delta\mathbf{v}$ is perpendicular to $\Delta\mathbf{r}$.

Since average acceleration is along Δ**v** **[a = Δv/Δt],**
The average acceleration **a** is perpendicular to Δ**r.** When $\Delta t \to 0$ the average acceleration becomes the instantaneous acceleration. It is directed towards the centre. Thus, we find that the acceleration of an object in uniform circular motion is always directed towards the centre of the circle
The magnitude of **a** is, by definition, given by,**a =** lim$\Delta t \to 0$ **(Δv/Δt)**
Let the angle between position vectors **r** and **r**' be $\Delta\theta$. Since the velocity vectors **v** and **v**' are always perpendicular to the position vectors, the angle between them is also $\Delta\theta$. Therefore, the triangle CPP' formed by the position vectors and the triangle GHI formed by the velocity vectors **v**, **v**' and Δ**v** are similar (a1). Therefore

fig(a)



$$\frac{I\Delta\mathbf{v}I}{v} = \frac{I\Delta rI}{\mathbf{R}}$$

$$I\Delta\mathbf{v}I = \mathbf{v}\,\frac{I\Delta rI}{\mathbf{R}}$$

Therefore, $IaI = \lim_{\Delta t \to 0} (\Delta\mathbf{v}/\Delta\mathbf{t})$

$$IaI = \lim_{\Delta t \to 0} \mathbf{v}\,\frac{I\Delta rI}{R\Delta t}$$

$$IaI = \frac{\mathbf{v}}{R} \lim_{\Delta t \to 0} \frac{I\Delta rI}{\Delta t}$$

$$.\ 0R \lim_{\Delta t \to 0} \frac{I\Delta rI}{\Delta t} = v$$

Therefore, the centripetal acceleration a_c is $a_c = \left[\frac{v}{R}\right] v = \frac{v^2}{R}$

Thus, the acceleration of an object moving with speed v in a circle of radius R has a magnitude v^2/R and is always **directed towards the centre**. This is why this acceleration is called **centripetal acceleration.**

As the object moves from P to P' in time Δt the line CP (Fig. a) turns through an angle $\Delta\theta$ as shown in the figure. $\Delta\theta$ is called angular distance. We define the angular speed ω as the time rate of change of angular displacement:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Now, if the distance travelled by the object during the time Δt is Δs, i.e. PP' is Δs, then :

$$v = \frac{\Delta s}{\Delta t}$$

but $\Delta s = R\,\Delta\theta$ Therefore:

$$v = R\,\frac{\Delta\theta}{\Delta t} = R\omega$$

$$v = R\,\omega$$

$$a_c = \frac{v^2}{R} = \frac{R^2\,\omega^2}{R} = R\,\omega^2$$

$$a_c = R\,\omega^2$$

We can express centripetal acceleration a_c in terms of angular speed:
The time taken by an object to make one revolution is known as its time period T and the number of revolution made in one second is called its frequency ν (=1/T). However, during this time the distance moved by the object is $s = 2\pi R$.
Therefore, $v = 2\pi R/T = 2\pi R\nu$
In terms of frequency ν, we have

$\omega = 2\pi\nu$

$v = 2\pi R\nu$

$ac = 4\pi^2\,\nu^2 R$

Problem An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s. (a) What is the angular speed, and the linear speed of the motion? (b) Is the acceleration vector a constant vector? What is its magnitude?